Math 565: Functional Analysis
Lecture 14

Thm. An infinite dimensional Banach space X doesn't admit a ctbl linear (Hamel) basis.

Proof. Otherwise, X is a ctbl union of finite dim subspaces, each of which is automatically nowhere dense, contradicting that X is nonmeagre. $\square$

Recall the slogan: weak regularity + algebraic structure $\Rightarrow$ strong regularity.

Open mapping thm. Let X be Banach spaces. Then any continuous ($\Leftrightarrow$ bdd) linear surjective $T: X \to Y$ is open, i.e. maps open sets to open sets.

Proof. We will repeatedly use the basic fact that translation is an isometry (hence maps open balls to open balls of the same radius) and dilation is a homeomorphism which maps open balls centered at 0 to open balls centered at 0. Denote by $B_r^X(x_0)$ and $B_r^Y(y_0)$ the open balls in X and Y of radius $r>0$ centered at $x_0 \in X$ and $y_0 \in Y$; omit writing x_0 and y_0 if they are 0.

Claim 1. It is enough to show that $T(B_1^X) \supseteq B_r^Y$ for some $r>0$.

Pf of Claim. We need to show that $T(U)$ is open for each $U \subseteq X$, equivalently, that each $y_0 \in T(U)$ has a ball $B_\varepsilon^Y(y_0) \subseteq T(U)$. Letting $x_0 \in U$ with $Tx_0 = y_0$, the openness of U gives $B_\delta^X(x_0) \subseteq U$ for some $\delta > 0$. But $B_\delta^X(x_0) = B_\delta^X - x_0$ and $T(B_\delta^X) \supseteq B_{\delta r}^Y$ by our assumption and dilation. Hence $T(B_\delta^X(x_0)) = T(B_\delta^X) - T(x_0) \supseteq B_{\delta r}^Y - y_0 = B_{\delta r}^Y(y_0)$. $\square$ (Claim)

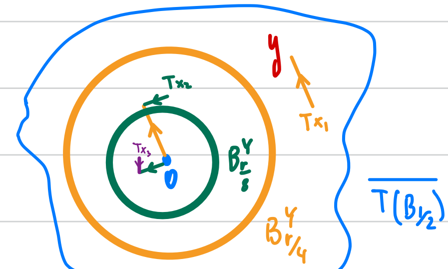
Claim 2. $\overline{T(B_1^X)} \supseteq B_r^Y$ for some $r>0$. Equivalently $\overline{T(B_\ell^X)} \supseteq B_{\ell r}^Y$ for all $\ell > 0$.

Pf of Claim. Since $X = \bigcup_{n \in \mathbb{N}} B_n^X$, surjectivity of T gives $Y = \bigcup_{n \in \mathbb{N}} T(B_n^X)$. Because Y is nonmeagre, $T(B_n^X)$ is somewhere dense for some $\mathbb{N}$, so $\overline{T(B_n^X)}$ contains a nonempty open set U. Take $y_0 \in T(B_n^X) \cap U$ so $y_0 = Tx_0$ for some $x_0 \in B_n^X$ and $B_\varepsilon^Y(y_0) \subseteq U$. Then

$$\overline{T(B_{2n}^X)} \supseteq \overline{T(B_n^X - x_0)} = \overline{T(B_n^X)} - T(x_0) \supseteq B_\varepsilon^Y(y_0) - y_0 = B_\varepsilon^Y.$$

By dilation, we have that $\forall \ell>0$, $\overline{T(B^X_\ell)} = \frac{\ell}{2n}\overline{T(B^X_{2n})} \supseteq \frac{\ell}{2n} B^Y_\varepsilon = B^Y_{\ell\cdot\varepsilon/2n}$, so $r := \frac{\varepsilon}{2n}$ works. □ (Claim)

Claim 3. $T(B_1) \supseteq \overline{T(B^X_{1/2})}$.



Pf of Claim. Fix $y\in\overline{T(B^X_{1/2})}$, so $\exists\, x_1\in B^X_{1/2}$ with $\|y-Tx_1\|<\frac{r}{4}$. Thus $y-Tx_1 \in B^Y_{r/4} \subseteq \overline{T(B^X_{1/4})}$, so $\exists\, x_2\in B^X_{1/4}$ with $\|y-Tx_1-Tx_2\|<\frac{r}{8}$. Thus, $y-Tx_1-Tx_2 \in B^Y_{r/8} \subseteq T(B^X_{1/2})$, etc. We obtain $(x_n)\subseteq X$ such that for each $n\geq 1$, $x_n\in B_{1/2^n}$ and $\|y-\sum_{i=1}^{n} Tx_i\| < \frac{r}{2^{n+1}}$.

In particular, $\|x_n\|<\frac{1}{2^n}$ so the series $\sum_{n\geq1} x_n$ converges absolutely, indeed $\sum_{n\geq1}\|x_n\| < \sum_{n\geq1}\frac{1}{2^n}=1$. By the completeness of X, $x=\sum_{n\geq1}x_n$ exists and $\|x\|\leq\sum_{n\geq1}\|x_n\| < \sum_{n\geq1}\frac{1}{2^n}=1$, so $x\in B^X_1$. By the continuity of T, we have $Tx = T(\sum_{n\geq1}x_n)=\sum_{n\geq1}Tx_n$ and by construction $\|y-\sum_{n\geq1}Tx_n\|=0$, so $\|y-Tx\|=0$, i.e. $y=Tx$. □ (Claim)

By Claim 3 and 2, $T(B_1)\supseteq\overline{T(B^X_{1/2})}\supseteq B^Y_{r/2}$, which finishes the proof by Claim 1. QED

Cor. Let X,Y be Banach spaces. Then any continuous linear bijection $T:X\to Y$ is a homeomorphism, i.e. T^{-1} is automatically continuous.

Cor. If $\|\cdot\|_1, \|\cdot\|_2$ are two complete norms on a vector space X and $\|\cdot\|_1 \leq C\|\cdot\|_2$ for some $C>0$, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are Lipschitz equivalent, i.e. we also have $\|\cdot\|_2\leq C'\|\cdot\|_1$ for some $C'>0$.

Proof. Apply the previous corollary to the identity map $(X,\|\cdot\|_1)\to(X,\|\cdot\|_2)$. □

Cor (First isomorphism theorem). Let X,Y be Banach spaces and $T:X\to Y$ be a continuous linear map. Then $X/_{\ker T}$ and $T(X)$ are isomorphic $\iff$ $T(X)$ is closed.

Proof. HW

Cor. Every separable Banach space X is isomorphic to a quotient $\ell^1(\mathbb{N})/Y$ by some closed subspace $Y\subseteq\ell^1(\mathbb{N})$.

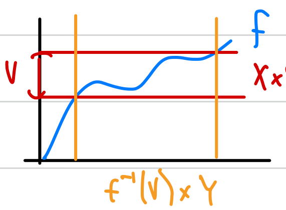
The next corollary of the open mapping theorem is the well-known closed graph theorem, for which we need to recall basic facts about continuous functions.

Fact 1. If X, Y are Hausdorff topological spaces and $f: X \to Y$ is continuous, then the graph $G_f := \{(x,y) \in X \times Y : f(x) = y\}$ is closed in $X \times Y$ (with the product top.)

Proof. Fix $(x,y) \in G_f^c$, so $f(x) \neq y$ hence $\exists$ disjoint open $V_0 \ni f(x)$ and $V_1 \ni y$ in Y. By continuity, $\exists$ open $U \ni x$ such that $f(U) \subseteq V_0$ so $(U \times V_1) \cap G_f = \emptyset$, so $(x,y) \in U \times V_1 \subseteq G_f^c$, hence G_f^c is open. $\square$

Fact 2 (surprizing). Let X, Y be arbitrary top. spaces. A function $f: X \to Y$ is continuous $\Leftrightarrow$ $\mathrm{proj}_X : G_f \to X$ is open $\Leftrightarrow$ $\mathrm{proj}_X : G_f \to X$ is homeomorphism.

Proof. The main observation is that for each $V \subseteq Y$, $(X \times V) \cap G_f = (f^{-1}(V) \times Y) \cap G_f$.



$\Leftarrow$. If $\mathrm{proj}_X : G_f \to X$ is open, then $\mathrm{proj}(X \times V)$ is open for each open $V \subseteq Y$, hence $f^{-1}(V) = \mathrm{proj}_X((f^{-1}(V) \times Y) \cap G_f)$ is open, hence f is continuous.

$\Rightarrow$. If f is continuous then $f^{-1}(V)$ is open for each open $V \subseteq Y$. So $\mathrm{proj}_X((X \times V) \cap G_f) = f^{-1}(V)$ is open. Also, $\mathrm{proj}_X((U \times Y) \cap G_f) = U$ which is open for each open $U \subseteq X$. Since the sets $X \times V$ and $U \times Y$ generate the top on $X \times Y$ and $\mathrm{proj}_X|_{G_f}$ is 1-1 (so it respects intersections), $\mathrm{proj}_X : G_f \to X$ is open. $\square$

Closed graph theorem. Let X, Y be Banach spaces. Then every linear $T: X \to Y$ whose graph is closed is continuous ($\Leftrightarrow$ bdd).

Proof. G_T being closed in the Banach space $X \times Y$ is a Banach space, so $\mathrm{proj}_X : G_T \to X$ is an open map, by the open mapping thm, hence T is continuous. $\square$

Remark. Why is the closed graph thm useful? It reduces proving the continuity of a linear map $T: X \to Y$ to proving the closedness of G_f, i.e. given $x_n \to x$ in X, instead of showing that (Tx_n) converges in Y and the limit is Tx, we can assume (Tx_n) converges and we only need to show that the limit is Tx.